Evaluation of Symmetric functions of roots.

Symmetric function: → When any two of the roots are interchanged of the given expression and then the equation remains unaltered, then such an expression is called a symmetric function of the roots of given equation.

i.e. $\alpha^2\beta+\alpha^2\delta+\beta^2\alpha+\beta^2\delta+\delta^2\alpha+\delta^2\beta$ is a symmetric function of the roots α, β, δ of a cubic equation.

Q: Find the values of the following symmetric functions for the cube equation.

$x^3+px^2+qx+r=0$ whose roots are α, β, γ (i) $\sum\alpha^2$ (ii) $\sum\alpha^2\beta^2$ (iii) $\sum\alpha^2\beta$ (iv) $\sum\alpha^3$ (v) $\sum\alpha^3\beta^3$ (vi) $\sum\frac{\beta^2+\gamma^2}{\beta\gamma}$ (vii) $\sum\frac{\beta^2+\gamma^2}{\beta+\gamma}$.

Ans:— Given that α, β, γ are the roots of the equation $x^2+px^2+qx+r=0$ then

$$\sum\alpha = \alpha+\beta+\gamma = -p$$
$$\sum\alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = q$$
$$\alpha\beta\gamma = r$$

We have (i) $(\sum\alpha)^2 = (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma$
$= \sum\alpha^2+2\sum\alpha\beta$

2.

$$\Rightarrow \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$\therefore \Sigma\alpha^2 = (-p)^2 - 2q = p^2 - 2q$$

(ii) $(\Sigma\alpha\beta)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$

$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta^2\gamma + 2\alpha^2\beta\gamma + 2\alpha\beta\gamma^2$$

$$\Rightarrow (\Sigma\alpha\beta)^2 = \Sigma\alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$\Rightarrow \Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\,\Sigma\alpha$$

$$= q^2 - 2(-r)(-p) = q^2 - 2pr$$

(iii) $\Sigma\alpha\,\Sigma\alpha\beta = (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$$= \alpha^2\beta + \alpha\beta\gamma + \gamma\alpha^2 + \alpha\beta^2 + \beta^2\gamma + \alpha\beta\gamma + \alpha\beta\gamma + \beta\gamma^2 + \gamma^2\alpha$$

$$= \Sigma\alpha^2\beta + 3\alpha\beta\gamma$$

$$\Rightarrow \Sigma\alpha^2\beta = \Sigma\alpha\cdot\Sigma\alpha\beta - 3\alpha\beta\gamma$$

$$= (-p)(q) - 3(-r) = -pr + 3r = 3r - pq$$

(iv) $\Sigma\alpha\,\Sigma\alpha^2 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2) = \Sigma\alpha^3 + \Sigma\alpha^2\beta$

$$\Rightarrow \Sigma\alpha^3 = \Sigma\alpha\,\Sigma\alpha^2 - \Sigma\alpha^2\beta = (-p)(p^2-2q) - (3r - pq)$$

$$= -p^3 + 2pq - 3r + pq = 3pq - 3r - p^3$$

(v) $\Sigma\alpha\beta\,\Sigma\alpha^2\beta^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)$

$$= \Sigma\alpha^3\beta^3 + \Sigma\alpha^3\beta^2\gamma = \Sigma\alpha^3\beta^3 + \alpha\beta\gamma\,\Sigma\alpha^2\beta$$

$$\therefore \Sigma\alpha^3\beta^3 = \Sigma\alpha\beta\,\Sigma\alpha^2\beta^2 - \alpha\beta\gamma\,\Sigma\alpha^2\beta$$

$$= q(q^2 - 2pr) - (-r)(3r - pq)$$

$$= q^3 - 2pqr + 3r^2 - pqr = q^3 - 3pqr + 3r^2$$

(vi) $$\Sigma\frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$= \frac{\alpha(\beta^2+\gamma^2) + \beta(\gamma^2+\gamma^2) + \gamma(\alpha^2+\beta^2)}{\alpha\beta\gamma}$$

$$\frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma} = \frac{3r - pq}{-r} = \frac{pq - 3r}{r}$$

(vii) $$\sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{\beta^2+\gamma^2}{\beta+\gamma} + \frac{\gamma^2+\alpha^2}{\gamma+\alpha} + \frac{\alpha^2+\beta^2}{\alpha+\beta}$$

$$= \frac{\sum(\alpha+\gamma)(\alpha+\beta)(\beta^2+\gamma^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}$$

$$= \frac{\sum(\beta^2+\gamma^2)(\alpha+\beta)(\alpha+\gamma)}{(\beta+\gamma)(\alpha\gamma+\beta\gamma+\alpha^2+\alpha\beta)}$$

$$= \frac{\sum(\beta^2+\gamma^2)\left[\alpha^2+(\alpha\gamma+\alpha\beta+\beta\gamma)\right]}{\sum\alpha^2\beta+2\alpha\beta\gamma}$$

$$= \frac{\alpha^2(\beta^2+\gamma^2)+q\sum(\beta^2+\gamma^2)}{(3r-pq)+2(-r)}$$

$$= \frac{2\sum\alpha^2\beta^2+2q\sum\alpha^2}{3r-pq-2r}$$

$$= \frac{2(q^2-2pr)+2q(p^2-2q)}{r-pq}$$

$$= \frac{2p^2q-4pr-2q^2}{r-pq}.$$ Ans.

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